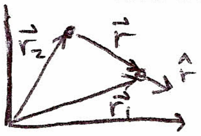
Phys 410
Spring 2013, Prof. Anlage
15 February, 2013

Problem 1. Both the Coulomb and gravitational forces lead to potential energies of the form $U(\vec{r}_1 - \vec{r}_2) = \gamma/|\vec{r}_1 - \vec{r}_2|$, where $\gamma = kq_1q_2$ for the Coulomb force and $\gamma = -Gm_1m_2$ for gravity, and $\vec{r}_1$, and $\vec{r}_2$ are the positions of the two particles. Show in detail that $-\nabla_1 U(\vec{r}_1 - \vec{r}_2)$ is the force on particle 1 and $-\nabla_2 U(\vec{r}_1 - \vec{r}_2)$ is that on particle 2.

Hint: Define $\vec{r} = \vec{r}_1 - \vec{r}_2$ which is a vector that points from particle 2 to particle 1. The PE is $U = \frac{\gamma}{r}$ and the forces can be written as $\vec{F}_{12} = \frac{\gamma}{r^2}\hat{r}$ and $\vec{F}_{21} = -\frac{\gamma}{r^2}\hat{r}$.



$$\vec{F} = -\vec{\nabla} U$$

Look at the x-component of $-\nabla_1 U$:

$$(-\vec{\nabla}_1 U)_x = \frac{-\partial U}{\partial x_1} = +\frac{\gamma}{r^2}\frac{\partial r}{\partial x_1}$$

with $r = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

so

$$\frac{\partial r}{\partial x_1} = \frac{\frac{1}{2}2(x_1-x_2)}{\sqrt{\cdots}} = \frac{x_1-x_2}{r}$$

Hence

$$(-\vec{\nabla}_1 U)_x = \frac{\gamma}{r^2}\frac{x_1-x_2}{r}.$$

Now repeat for the y and z components of $-\vec{\nabla}_1 U$

$$-\vec{\nabla}_1 U = \frac{\gamma}{r^3}\left[(x_1-x_2)\hat{x} + (y_1-y_2)\hat{y} + (z_1-z_2)\hat{z}\right] = \frac{\gamma}{r^3}(\vec{r}_1 - \vec{r}_2) = \frac{\gamma}{r^3}\vec{r}$$

$$= \frac{\gamma}{r^2}\hat{r} = \vec{F}_{12}$$

Now evaluate $(-\vec{\nabla}_2 U)_x$. The only difference will be in the chain rule derivative

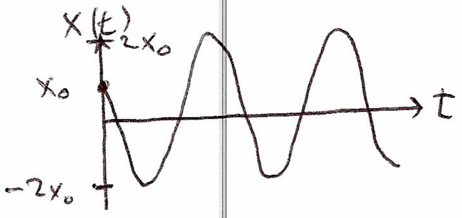
$$\frac{\partial r}{\partial x_2} = \frac{\frac{1}{2}2(x_1-x_2)(-1)}{\sqrt{\cdots}} = \frac{-(x_1-x_2)}{r}$$

This new minus sign will propagate through to become

$$-\vec{\nabla}_2 U = \frac{-\gamma}{r^3}(\vec{r}_1 - \vec{r}_2) = -\frac{\gamma}{r^3}\vec{r} = \frac{-\gamma}{r^2}\hat{r} = \vec{F}_{21}$$

Problem 2

A mass on the end of a spring is oscillating with angular frequency ω. At $t = 0$ its position is $x_0 > 0$ and I give it a kick so that it moves back toward the origin and executes simple harmonic motion with amplitude $2x_0$. Find the position as a function of time in the form $x(t) = A\cos(\omega t - \delta)$.



$x(0) = x_0$

$\dot{x}(0) < 0$

We are told that $x(t) = A\cos(\omega t - \delta)$ with $A = 2x_0$

At $t = 0$ we have $x(0) = x_0 = A\cos(-\delta) = 2x_0\cos(-\delta)$

or $\cos\delta = 1/2$. Hence $\delta = \pm\pi/3$.

Now look at the velocity $\dot{x}(t) = -A\omega\sin(\omega t - \delta)$, so that $\dot{x}(0) = -A\omega\sin(-\delta) = +2x_0\omega\sin\delta$. All we know about $\dot{x}(0)$ is that it is negative. Hence it must be that $\delta = -\pi/3$.

The full solution is $x(t) = 2x_0\cos(\omega t + \pi/3)$